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TETRAHEDRON EQUATIONS, BOUNDARY STATES AND HIDDEN STRUCTURE OF $\mathscr{U}_q(D_n^{(1)})$

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ABSTRACT. Simple periodic $3d \to 2d$ compactification of the tetrahedron equations gives the Yang-Baxter equations for various evaluation representations of $\mathscr{U}_q(\widehat{sl}_n)$. In this paper we construct an example of fixed non-periodic $3d$ boundary conditions producing a set of Yang-Baxter equations for $\mathscr{U}_q(D_n^{(1)})$. These boundary conditions resemble a fusion in hidden direction.

The tetrahedron equation can be viewed as a local condition providing existence of an infinite series of Yang-Baxter equations. In the applications to quantum groups the method of tetrahedron equation is a powerful tool for generation of R-matrices and L-operators for various "higher spin" evaluation representations. This has been demonstrated in [1] for $\mathscr{U}_q(\widehat{sl}_n)$ and in [4] for super-algebras $\mathscr{U}_q(\widehat{gl}_{n|m})$.

The main principle producing the cyclic $\widehat{sl}_n$ structure is the trace in hidden "third" direction. In this paper we introduce another boundary condition, a certain boundary states still providing the existence of effective Yang-Baxter equation and integrability.

We shall start with a short remained of a (super-)tetrahedron equation and $\widehat{sl}_n$ compactification in their elementary form. The simplest known tetrahedron equation in the tensor product of six spaces $B_1 \otimes F_2 \otimes \cdots \otimes F_5 \otimes B_6$ is

$$\mathsf{R}_{B_1F_2F_3}\mathsf{R}_{B_1F_4F_5}\mathsf{R}_{F_2F_4B_6}\mathsf{R}_{F_3F_5B_6} \;=\; \mathsf{R}_{F_3F_5B_6}\mathsf{R}_{F_2F_4B_6}\mathsf{R}_{B_1F_4F_5}\mathsf{R}_{B_1F_2F_3}\;, \tag{1}$$

where $F_i = \{|0\rangle, |1\rangle\}_i$ is a representation space of Fermi oscillator

$$\boldsymbol{f}^+|0\rangle \;=\; |1\rangle\;,\quad \boldsymbol{f}^-|1\rangle = |0\rangle\;. \tag{2}$$

Odd operators $\boldsymbol{f}_i^{\pm}$ in different components i of their tensor product anti-commute and $(\boldsymbol{f}_i^{\pm})^2 = 0$. It is convenient to introduce projectors

$$\boldsymbol{M}_i = \boldsymbol{f}_i^+\boldsymbol{f}_i^-\;,\quad \boldsymbol{M}_i^0 = \boldsymbol{f}_i^-\boldsymbol{f}_i^+\;,\quad [\boldsymbol{f}_i^+, \boldsymbol{f}_i^-]_+ = \boldsymbol{M}_i^0 + \boldsymbol{M}_i = 1\;. \tag{3}$$

Operator $\boldsymbol{M}_i^0$ is the projector to vacuum, $\boldsymbol{M}_i$ is the occupation number and $\boldsymbol{M}^0\boldsymbol{M} = 0$.

Space B_i stands for representation space of i-th copy of q-oscillator,

$$\boldsymbol{b}^+\boldsymbol{b}^- = 1 - q^{2\boldsymbol{N}}\;,\quad \boldsymbol{b}^-\boldsymbol{b}^+ = 1 - q^{2\boldsymbol{N}+2}\;,\quad q^{\boldsymbol{N}}\boldsymbol{b}^{\pm} = \boldsymbol{b}^{\pm}q^{\boldsymbol{N}\pm 1}\;. \tag{4}$$

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In this paper we imply the unitary Fock space representation, $(\boldsymbol{b}^-)^\dagger = \boldsymbol{b}^+$, defined by

$$(5)\qquad \boldsymbol{N}|n\rangle = |n\rangle n\;,\quad \boldsymbol{b}^-|0\rangle = 0\;,\quad |n\rangle \;=\; \frac{\boldsymbol{b}^{+n}}{\sqrt{(q^2;q^2)_n}}\,|0\rangle\;,\quad n\geq 0\;,$$

where $(x;q^2)_n \;=\; (1-x)(1-q^2x)\cdots(1-q^{2n-2}x)$. In terms of creation, annihilation and occupation number operators the R-matrices in (1) are given [4] by

$$(6)\qquad \mathsf{R}_{B_1F_2F_3} = \boldsymbol{M}_2^0\boldsymbol{M}_3^0 - q^{\boldsymbol{N}_1+1}\boldsymbol{M}_2\boldsymbol{M}_3^0 + q^{\boldsymbol{N}_1}\boldsymbol{M}_2^0\boldsymbol{M}_3 - \boldsymbol{M}_2\boldsymbol{M}_3 + \boldsymbol{b}_1^-\boldsymbol{f}_2^+\boldsymbol{f}_3^- - \boldsymbol{b}_1^+\boldsymbol{f}_2^-\boldsymbol{f}_3^+$$

and

$$(7)\qquad \mathsf{R}_{F_1F_2B_3} = \boldsymbol{M}_1^0\boldsymbol{M}_2^0 + \boldsymbol{M}_1\boldsymbol{M}_2^0q^{\boldsymbol{N}_1+1} - \boldsymbol{M}_1^0\boldsymbol{M}_2q^{\boldsymbol{N}_1} - \boldsymbol{M}_2\boldsymbol{M}_3 + \boldsymbol{f}_1^+\boldsymbol{f}_2^-\boldsymbol{b}_3^- - \boldsymbol{f}_1^-\boldsymbol{f}_2^+\boldsymbol{b}_3^+\;.$$

Both operators R are unitary roots of unity. The constant tetrahedron equation (1) can be verified in the operator language straightforwardly.

Define next the "monodromy" of R-matrices as the ordered product

$$(8)\qquad \mathsf{R}_{\Delta_n(B_1F_2),F_3} \;=\; \mathsf{R}_{B_{1:1}F_{2:1}F_3}\mathsf{R}_{B_{1:2}F_{2:2}F_3}\cdots\mathsf{R}_{B_{1:n}F_{2:n}F_3} \;\leftrightharpoons\; \overset{\curvearrowright}{\prod_{j=1..n}}\mathsf{R}_{B_{1:j}F_{2:j}F_3}\;.$$

Here the convenient "co-product" notation stands for a tensor power of corresponding spaces,

$$(9)\qquad \Delta_n(B_1) \;=\; \bigotimes_{j=1}^{n} B_{1:j}\;,\quad \Delta_n(F_2) \;=\; \bigotimes_{j=1}^{n} F_{2:j}\;.$$

The repeated use of (1) provides

$$(10)\qquad \begin{aligned}&\mathsf{R}_{\Delta_n(B_1F_2),F_3}\mathsf{R}_{\Delta_n(B_1F_4),F_5}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\mathsf{R}_{F_3F_5B_6}\\ &\qquad\qquad = \;\mathsf{R}_{F_3F_5B_6}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\mathsf{R}_{\Delta_n(B_1F_4),F_5}\mathsf{R}_{\Delta_n(B_1F_2),F_3}\;.\end{aligned}$$

Note the conservation laws:

$$(11)\qquad v^{-\boldsymbol{M}_3}u^{-\boldsymbol{M}_5}\left(\frac{u}{v}\right)^{\boldsymbol{N}_6}\mathsf{R}_{F_3F_5B_6} \;=\; \mathsf{R}_{F_3F_5B_6}v^{-\boldsymbol{M}_3}u^{-\boldsymbol{M}_5}\left(\frac{u}{v}\right)^{\boldsymbol{N}_6}\;.$$

Multiplying (10) by the u,v-term in $F_3\otimes F_5\otimes B_6$ and by $\mathsf{R}^{-1}_{F_3F_5B_6}$, and making then the traces over $F_3\otimes F_5\otimes B_6$, we come to the Yang-Baxter equation

$$(12)\qquad L_{\Delta_n(B_1F_2)}(v)L_{\Delta_n(B_1F_4)}(u)R_{\Delta_n(F_2F_4)}(u/v) \;=\; R_{\Delta_n(F_2F_4)}(u/v)L_{\Delta_n(B_1F_4)}(u)L_{\Delta_n(B_1F_2)}(v)\;,$$

where

$$(13)\qquad L_{\Delta_n(B_1F_2)}(v) = \underset{F_3}{\mathrm{Str}}\left(v^{-\boldsymbol{M}_3}\mathsf{R}_{\Delta_n(B_1F_2),F_3}\right)\;,\quad R_{\Delta_n(F_2F_4)}(w) = \underset{B_6}{\mathrm{Tr}}\left(w^{\boldsymbol{N}_6}\mathsf{R}_{\Delta_n(F_2F_4),B_6}\right)\;.$$

This is the case of $\mathscr{U}_q(\widehat{sl}_n)$. Two-dimensional R-matrices (13) have the centers

$$(14)\qquad J_i = \sum_{j=1}^{n}\boldsymbol{M}_{i:j}\quad\text{for fermions and}\quad J_1 = \sum_{j=1}^{n}\boldsymbol{N}_{1:j}\quad\text{for bosons.}$$

Irreducible components of R-matrices and L-operators (13) correspond to fixed values of J_i. In particular, $\Delta_n(F)$ is the sum of all antisymmetric tensor representations of sl_n,

$$\dim \Delta_n(F) \;=\; 2^n \;=\; \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\;. \tag{15}$$

The Dirac spinor representation of D_n has the same dimension 2^n, it is the direct sum of two irreducible Weyl spinors with dimensions 2^{n-1}. It is evident intuitively, the structure of D_n will appear if the total occupation number J of $\Delta_n(F)$ is not a center of L-operators and R-matrices, but all operators preserve just the parity of J. Also, since the dimension of vector representation of D_n is $2n$, we need to double the number of bosons.

Consider now two copies of (1) and further of (10) glued in the "second" direction. This consideration keeps the desired space $\Delta_n(F)$ and doubles the number of bosons. The repeated use of (1) provides

$$\begin{aligned}&\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)}\mathsf{R}_{\Delta(B_1)F_4\Delta(F_5)}\mathsf{R}_{F_2F_3B_6}\mathsf{R}_{\Delta'(F_3F_5)B_6}\\ &\qquad\qquad = \mathsf{R}_{\Delta'(F_3F_5)B_6}\mathsf{R}_{F_2F_3B_6}\mathsf{R}_{\Delta(B_1)F_4\Delta(F_5)}\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)},\end{aligned} \tag{16}$$

where

$$\mathsf{R}_{\Delta(B_1)F_2\Delta(F_3)} \;=\; \mathsf{R}_{B_1F_2F_3}\mathsf{R}_{B_1'F_2F_3'} \quad\text{and}\quad \mathsf{R}_{\Delta'(F_3F_5)B_6} = \mathsf{R}_{F_3'F_5'B_6}\mathsf{R}_{F_3F_5B_6}\;. \tag{17}$$

The key observation is the existence of a family of eigenvectors of operator $\mathsf{R}_{\Delta'(F_3F_5)B_6}$:

$$\mathsf{R}_{\Delta'(F_3F_5)B_6}|\psi_{\Delta(F_3)}(v)\psi_{\Delta(F_5)}(u)\psi_{B_6}(u/v)\rangle \;=\; |\psi_{\Delta(F_3)}(v)\psi_{\Delta(F_5)}(u)\psi_{B_6}(u/v)\rangle\;, \tag{18}$$

where

$$\Delta(F) = F'\otimes F\;,\quad |\psi_{\Delta(F)}(v)\rangle \;=\; (1+v^{-1}\boldsymbol{f}^{+\prime}\boldsymbol{f}^{+})|0\rangle\;, \tag{19}$$

and in the unitary basis (3)

$$\langle 2k+1|\psi_B(w)\rangle = 0\;,\quad \langle 2k|\psi_B(w)\rangle = w^k\sqrt{\frac{(q^{4k+4};q^4)_\infty}{(q^{4k+2};q^4)_\infty}}\;. \tag{20}$$

The normalization of ψ_B is given by

$$\langle\overline{\psi}_B(w)|(\boldsymbol{b}^{\pm})^{2m}|\psi_B(w)\rangle \;=\; w^m\frac{(q^{2+4m}w^2;q^4)_\infty}{(w^2;q^4)_\infty}\;. \tag{21}$$

Considering now a length-n chain of (16) in the "third" direction and applying vectors $\psi_{\Delta(F_3)}(u)$, $\psi_{\Delta(F_5)}(v)$ and $\psi_B(u/v)$, we come to the Yang-Baxter equation

$$\begin{aligned}&L_{\Delta_n(\Delta(B_1)F_2)}(v)L_{\Delta_n(\Delta(B_1)F_4)}(u)R_{\Delta_n(F_2F_4)}(u/v)\\ &\qquad\qquad = R_{\Delta_n(F_2F_4)}(u/v)L_{\Delta_n(\Delta(B_1)F_4)}(u)L_{\Delta_n(\Delta(B_1)F_2)}(v)\end{aligned} \tag{22}$$

without trace construction:

$$L_{\Delta_n(\Delta(B_1)F_2)}(v) \;=\; \langle\overline{\psi}_{\Delta(F_3)}(v)|\mathsf{R}_{\Delta_n(\Delta(B_1)F_2),\Delta(F_3)}|\psi_{\Delta(F_3)}(v)\rangle \tag{23}$$

and

$$R_{\Delta_n(F_2F_4)}(w) \;=\; \langle\overline{\psi}_{B_6}(w)|\mathsf{R}_{\Delta_n(F_2F_4),B_6}|\psi_{B_6}(w)\rangle\;. \tag{24}$$

Matrix elements of $R_{\Delta_n(F_2F_4)}(w)$ can be calculated with the help of (21) and similar identities. The invariants of L-operator (23) and R-matrix (24) are: the parity of $J_2 = \sum \boldsymbol{M}_{2:j}$, similar parity of J_4 and

$$J_1 \;=\; \sum_{j=1}^{n}(\boldsymbol{N}_{1:j}-\boldsymbol{N}'_{1:j})\;. \tag{25}$$

A choice of different spectral parameters in bra- and ket-vectors in (23,24) is equivalent to the choice of equal spectral parameters by means of a gauge transformation.

The structure of D_n representation ring can be verified explicitly by a direct calculation of matrix elements of R-matrix (24) for small n and check of factor powers of $\det(\lambda - R)$.

As to $2n$-bosons space, irreducible components of $\Delta_n(\Delta(B_1))$ are in general infinite dimensional. However, a choice of Fock and anti-Fock space representations, Spectrum$(\boldsymbol{N}_{1:j}) = 0,1,2,\ldots$ and Spectrum$(\boldsymbol{N}'_{1:j}) = -1,-2,-3,\ldots$, makes $\Delta_n(\Delta(B_1))$ a direct sum of symmetric tensors of $O(2n)$.

The main result of this paper is a step forward to a classification of *integrable boundary conditions* in three-dimensional models. At least two scenarios are hitherto known: quasi-periodic boundary condition (13) and the boundary states condition (23,24). These conditions can be imposed for a layer-to-layer transfer matrix in different directions independently. In both scenarios the spectral parameters of effective two-dimensional models reside the boundary. Also, the boundary admits twists making the quantum groups classification inapplicable [3]. It worth noting one more possible scenario of integrable boundary conditions: yet unknown $3d$ reflection operators satisfying the tetrahedron reflection equations [2].

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References

[1] V. V. Bazhanov and S. M. Sergeev, *Zamolodchikov's tetrahedron equation and hidden structure of quantum groups*, J. Phys. A **39** (2006), no. 13, 3295–3310

[2] A. P. Isaev and P. P. Kulish, *Tetrahedron reflection equations*, Modern Phys. Lett. A **12** (1997), no. 6, 427–437

[3] S. Sergeev, *Ansatz of Hans Bethe for a two-dimensional lattice Bose gas*, J. Phys. A **39** (2006), no. 12, 3035–3045

[4] S. M. Sergeev, *Super-tetrahdera and super-algebras*, arXiv:0805.4653, 2008.

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